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Original Research

# Mathematical Modeling of Blood Flow Dynamics in the Cardiovascular System: Assumptions, Considerations, and Simulation Results

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**Abstract:**

The study of fluid dynamics is crucial to understanding fluid flow in the human body, and cardiovascular physics places a lot of concentration on blood flow modelling. Nevertheless, the models that have been created thus far with three-dimensional analysis are extremely complex. This research work offers study of blood pressure, flow and other flow-related parameters. Modelling artery was an extensible circular pipe with oscillating blood flow. Understanding factors that could lead to high blood pressure as one of the reasons for studying blood flow. The cardiovascular system equation is a straightforward differential equation that was developed under certain assumptions using Navier-Stokes equations. Generic study of normal blood flow was then created by applying some assumptions to the equation for the cardiovascular system. Poisuelli's equation was then used to extend this model to account for normal blood pressure. Upon completion of this study, an analysis was conducted to ascertain the validity of the suggested problem. According to the analysis, the model is able to account for various blood pressure and other flow characteristics of blood.

**Keywords:** Mathematical modeling, Equations describing cardiovascular system, Blood flow dynamics, Non-newtonian fluid behaviour Differential equations, Pressure regulation in circulatory system.



## Introduction:

A mathematical model elucidates the dynamics of a system within the physical domain through mathematical formulations. This process, termed mathematical modeling or modeling, entails the formulation of equations and relationships that represent the dynamics of the system under scrutiny. The cardiovascular system functions as the conduit for blood distribution throughout the organism. Comprising blood, heart, and blood vessels, the circulation of blood is orchestrated to sustain bodily functions. As blood traverses' vessels, it imparts force against vessel walls, resulting in blood pressure. The magnitude of blood pressure hinges primarily upon factors such as flow rate, vessel dimensions, and pressure differentials. Notably, the cardiovascular system encompasses three principles kind of blood vessels, capillaries, arteries, and veins. Arteries, which serve as conduits for blood egress from the heart, distribute blood to all body regions [16,15]. Arterioles, which branch from arteries, further subdivide into diminutive vessels known as capillaries, which act as conduits connecting the venous and arterial circulatory systems. Veins, which constitute a low-pressure network, facilitate the return of oxygen-depleted blood to the heart. All vessels are assumed to possess uniform properties, excluding variations in cross-sectional area size and length. Numerous scholarly works have delved into the intricacies of the cardiovascular system. A substantial body of literature explores functional imaging techniques and methodologies pertinent to heart function. Additional investigations have scrutinized aspects such as electrical activity, mechanical deformation, flow dynamics, fiber orientation, and cardiac modeling. This study contributes to the body of knowledge by presenting a study of blood flow and offering simulation outcomes based on said model [13,23]. To formulate a flow and pressure model, several assumptions have been postulated. Despite the role of the lungs in oxygenation, blood properties are assumed to be unaffected by oxygenation [2,4,8]. Moreover, it is posited that blood exhibits axial and radial flow solely in the z-direction within a three-dimensional framework, effectively eliminating the x- and y-directional components. Numerous studies have investigated various aspects of cardiovascular function, including functional imaging of the heart and modeling of its mechanical and electrical activity. This paper contributes by introducing a mathematical model of blood flow and analyzing simulation outcomes derived from this model [17,21,24]. Assumptions are established regarding the characteristics of blood vessels, blood properties, and flow dynamics to enable the construction of the model.

Understanding the dynamics of blood flow within this system is crucial for comprehending cardiovascular health and disease. Mathematical modeling is a powerful tool for investigating the intricacies of blood flow dynamics. By representing the cardiovascular system using mathematical language, researchers can gain insights into its behavior under various conditions and interventions. This paper aims to elucidate the process of mathematical modeling applied to the cardiovascular system, focusing on blood flow dynamics and pressure regulation. Components of the Cardiovascular System: The cardiovascular system comprises three main components: blood, heart, and blood vessels. Blood, a fluid medium containing cells and plasma, serves as a carrier of oxygen, nutrients, and waste products [3,5]. The heart, a muscular organ, acts as a pump to propel blood through vessels. Blood vessels, including capillaries, veins and arteries, form the conduit through which blood travels to different parts of the body and heart. Arteries serve as sizable, flexible conduits responsible for transporting oxygen-enriched blood from the heart to diverse tissues and organs.

Arterioles, smaller branches of arteries, further divide into capillaries, which facilitate the exchange of gases, nutrients, and waste products with surrounding tissues [10,12,18]. Veins, on the other hand, collect oxygen-poor blood from tissues and return it to the heart for reoxygenation. It is influenced by several factors, including the rate of blood flow, the diameter of blood vessels, and the pressure gradient along the vascular network. Variations in blood pressure can have significant implications for overall cardiovascular health, with high blood pressure (hypertension) being a major risk factor for cardiovascular disease [7,22]. To formulate a mathematical representation of blood flow dynamics, certain assumptions are made regarding the properties of blood vessels and blood flow. Arteries are considered cylindrical, deformable structures having circular cross-sections that are capable of changing in size in response to changes in blood flow. Blood is treated as a Newtonian fluid and is represented by the Navier-Stokes equation, which describes the motion and conservation of blood mass [15,16]. Mathematical modeling is a powerful tool used to describe real-world systems via mathematical language. In the context of the cardiovascular system, which serves as the body's blood distribution network, the system comprises three main components: blood, heart, and blood vessels.

Blood pressure, detected on vessel walls as blood flows through them, is influenced primarily by factors such as flow rate, vessel size, and pressure gradient [19,20,12]. There is a wealth of research in this domain, encompassing studies on functional heart imaging, electrical activity measurement, flow dynamics, and heart modeling. This paper contributes by presenting a mathematical blood flow model, alongside simulation outcomes derived from this model. The development of a blood flow and pressure model is grounded in several key assumptions. These assumptions encompass blood vessels as deformable cylinders having circular cross-sections that change in size in response to flow of blood [7,9,20]. Additionally, blood is modeled as a Newtonian fluid by the Navier-Stokes equation and continuity equation. Despite its dependence on oxygen supplied by the lungs, blood properties remain unaffected. Additionally, flow is assumed to exhibit both radial and axial motion in a single direction. In a three-dimensional framework, blood flow is predominantly oriented along the z-direction, while the effects of the remaining directional components are considered negligible.

**Mathematical Modelling and Formulation:**

Mathematical modeling plays a crucial role in understanding complex systems in the real world. This paper explores mathematical modelling of blood flow dynamics within the cardiovascular system. The cardiovascular system serves as the body's blood distribution network and comprise three main components: blood, heart, and blood vessels. Blood pressure, a key parameter in cardiovascular health, depends on factors such as flow rate, vessel size, and pressure gradient [17,20]. The system includes arteries, capillaries, and veins, each of which play a distinct role in blood circulation. The $u$, $v$ and $w$ are velocity components, in the $x$, $y$, and $z$ directions respectively, where ρ represents density, P represents pressure and μ signifies kinematic viscosity of the blood. The Navier-Stokes equation in cartesian coordinates, neglecting the influence of gravity orientation within the body, is expressed as follows [5,11]:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{2}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{3}$$

By disregarding velocity (tangential), a transformation of variables applied to cartesian equations results in the following set of expressions [15]:

$$[\frac{\partial w}{\partial t}+f\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right)] \tag{4}$$

$$[\frac{\partial f}{\partial t}+f\frac{\partial f}{\partial t}+w\frac{\partial f}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\partial^2 f}{\partial z^2}-\frac{f}{r^2}\right)] \tag{5}$$

$$[\frac{1}{r}\frac{\partial}{\partial r}(rf)+\frac{\partial w}{\partial z}=0] \tag{6}$$

In this context, let f(r,z,t) denote the radial flow component and w(r,z,t) represent the axial flow component in the z direction. The continuity equation is expressed as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho w)}{\partial z}=0 \tag{7}$$

$\gamma$ is a new variable, where $\gamma=\frac{r}{R(z,t)}$ and $R(z,t)$ is the radius of the artery. Moreover, the velocity profile is in polynomial form: [16]

$$w(\gamma,z,t)=\sum_{k=1}^{N}q_k(\gamma^{2k}-1) \tag{8}$$

In this scenario, q(z,t) represents the variable to be subsequently determined. For simplification, let N=1. Then,

$$w(\gamma,z,t)=q(z,t)(\gamma^2-1) \tag{9}$$

$$f(\gamma,z,t)=\gamma\frac{\partial R}{\partial z}f+\gamma\frac{\partial R}{\partial t}-\frac{\gamma}{N}\frac{\partial R}{\partial t}\sum_{k=1}^{N}\frac{1}{k}(\gamma^{2k}-1) \tag{10}$$

$$f(\gamma,z,t)=\frac{\partial R}{\partial z}\gamma f+\frac{\partial R}{\partial t}\gamma-\frac{\partial R}{\partial t}\gamma(\gamma^{2k}-1) \tag{11}$$

By utilizing governing equations,

$$[\frac{\partial q}{\partial t}-\frac{4q}{R}\frac{\partial R}{\partial t}-\frac{2q^2}{R}\frac{\partial R}{\partial z}+\frac{4\mu}{R^2}q+\frac{1}{\rho}\frac{\partial P}{\partial z}=0] \tag{12}$$

$$2\frac{\partial R}{\partial t}+\frac{R}{2}\frac{\partial q}{\partial z}+q\frac{\partial R}{\partial z}=0 \tag{13}$$

$$S=\pi R^2 \tag{14}$$

$$Q=\iint w\,\partial\gamma=\frac{1}{2}q\pi R^2 \tag{15}$$

From (14) and (15), $\frac{\partial q}{\partial t}$, $\frac{\partial q}{\partial z}$, $\frac{\partial R}{\partial t}$, and $\frac{\partial R}{\partial z}$ can be found.

By substituting $\frac{\partial q}{\partial t}$, $\frac{\partial q}{\partial z}$, $\frac{\partial R}{\partial t}$, and $\frac{\partial R}{\partial z}$ into (12), (13), and (13), we obtained,

$$\frac{\partial Q}{\partial t}+\frac{3Q}{S}\frac{\partial Q}{\partial z}-\frac{2Q^2}{S^2}\frac{\partial S}{\partial z}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z} \tag{16}$$

$$\frac{\partial S}{\partial t}+\frac{\partial Q}{\partial z}=0 \tag{17}$$

$$\frac{\partial Q}{\partial t}-\frac{3Q}{S}\frac{\partial S}{\partial t}-\frac{2Q^2}{S^2}\frac{\partial S}{\partial z}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0 \tag{18}$$

Eq. (18) is designated the 'master equation' going forward. Models for flow rate and pressure are obtained by implementing certain assumptions to this main equation, as detailed. In formulating the flow of the blood model, the cross-sectional area of the vessel is considered to be constant. Additionally, it is assumed to remain spatially constant, with no variations with axial distance. Furthermore, the pressure gradient is hypothesized to exhibit uniformity over the entirety of the distance under consideration. Equation (18) can be written as:

$$\frac{\partial Q}{\partial t}+\frac{4\pi\mu}{S}Q+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0 \tag{19}$$

This constitutes a mathematical model that simplifies a system or phenomenon by focusing on just a single spatial dimension-one for representation of the blood flow rate. The boundary conditions necessary for a solution, along with parameter values, can be derived from prior literature within this domain. Poiseuille's equation is employed to formulate mathematical model. Poiseuille's equation defines the relationship between the flow rate and pressure, expressed below:

$$Q=\frac{\pi R^4}{8L\mu}P \tag{20}$$

After inserting (45) into (44), a new equation is obtained as follows:

$$\frac{\pi R^4}{8L\mu}\frac{dP}{dt}+\frac{4\pi\mu}{S}\frac{\pi R^4}{8L\mu}P+\frac{S}{2\rho}\frac{\partial P}{\partial z}=0$$

$$\frac{dP}{dt}+\frac{4\mu}{R^2}P+\frac{4L\mu}{\rho R^2}\frac{\partial P}{\partial z}=0 \tag{21}$$

Equation (46) serves as a mathematical representation of blood pressure within the physiological system. The necessary boundary conditions and parameter values essential for solving this equation can be acquired from prior research conducted within this scientific discipline.

Pressure gradient, $\frac{\partial P}{\partial z}=100$ to40 $mmHg$

Kinematic viscosity of blood,$\mu=0.0035\ cm^2/S$

Density of blood, $\rho=1.043$ to1.057 $g/cm^3$

**Results and Discussion:**

In this study, a validation analysis was conducted on the proposed model. The equation describing the flow rate of blood was simplified in MATLAB. Fig. 1 shows the results for the cross-sectional areas. An important observation from this plot is the direct correlation between flow rate and area, suggesting that an increase in cross-sectional area leads to a greater blood flow rate. This observation aligns with the trends illustrated in Fig. 2. In the solution is depicted for varying pressure in Fig. 3. As illustrated, there is a discernible pressure differential along the length of the vessel, with higher pressures observed at the vessel's outset than at its terminus, thus establishing a pressure gradient. Notably, a heightened pressure gradient serves as a driving force for blood flow through vessels. Consistent with established principles [18], the greater the pressure gradient is, the more pronounced the rate of flow. Moreover, Fig. 3 demonstrates that under a constant pressure gradient, the blood flow rate diminishes over time. Conversely, an increase in the pressure gradient results in an increase in the blood flow rate. Blood pressure is formulated by equation (21) and can be resolved by MATLAB. Fig. 4 shows the results obtained for various cross-sectional areas. Notably, the plot reveals a decrease in blood pressure as the cross-sectional area increases. This observation underscores the relationship between blood pressure and cross-sectional area, which is consistent with the trends depicted in Figure 5 derived from Poiseuille's equation.

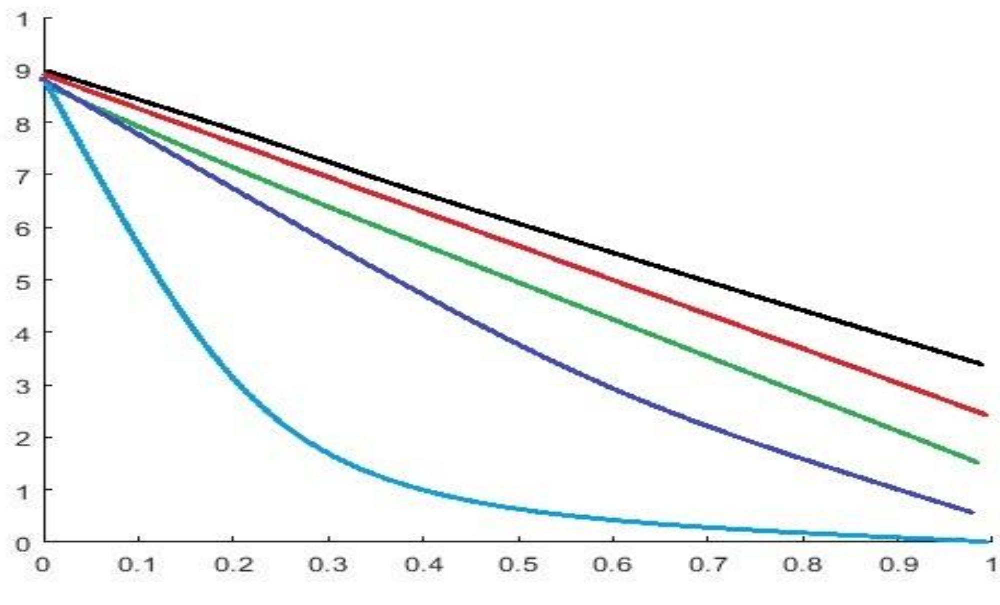


**Fig. 1. The flow rate varies with the cross-sectional area of the blood vessel.**

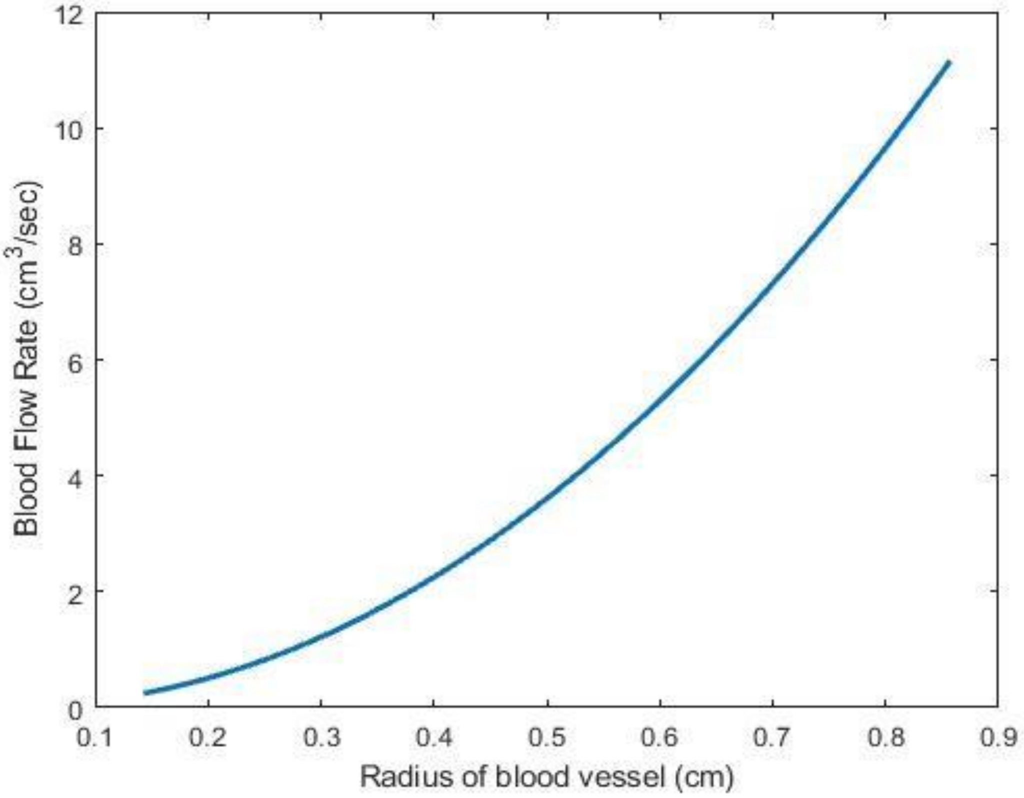


**Fig. 2. Illustration of the relationship between the flow rate and vessel radius.**

The solution is depicted for blood vessels of varying lengths by Fig.6. This study, revealed a notable increase in blood pressure concomitant with an increase in vessel length. The observed pattern of highest pressure at the vessel's outset and lowest pressure at its terminus is consistent with expectations.

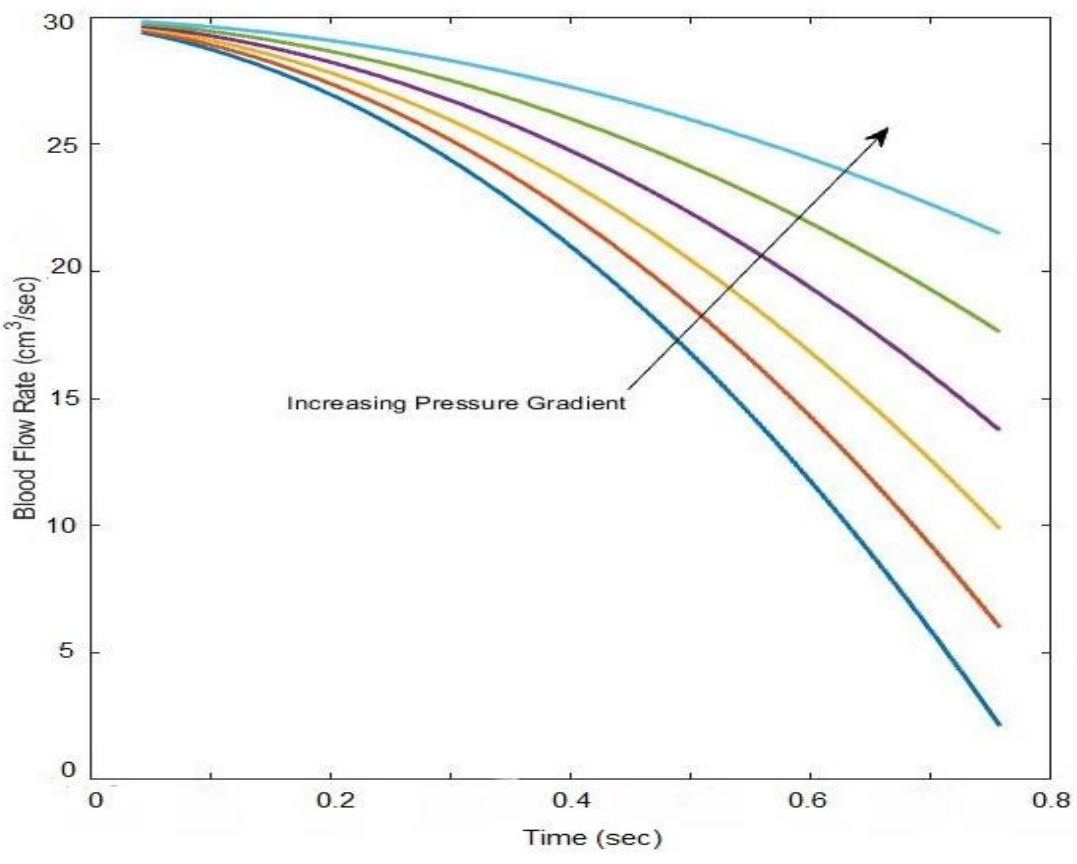


**Fig. 3. The blood flow rate for different pressure gradients ranging from 40 to 200 mmHg is shown.**

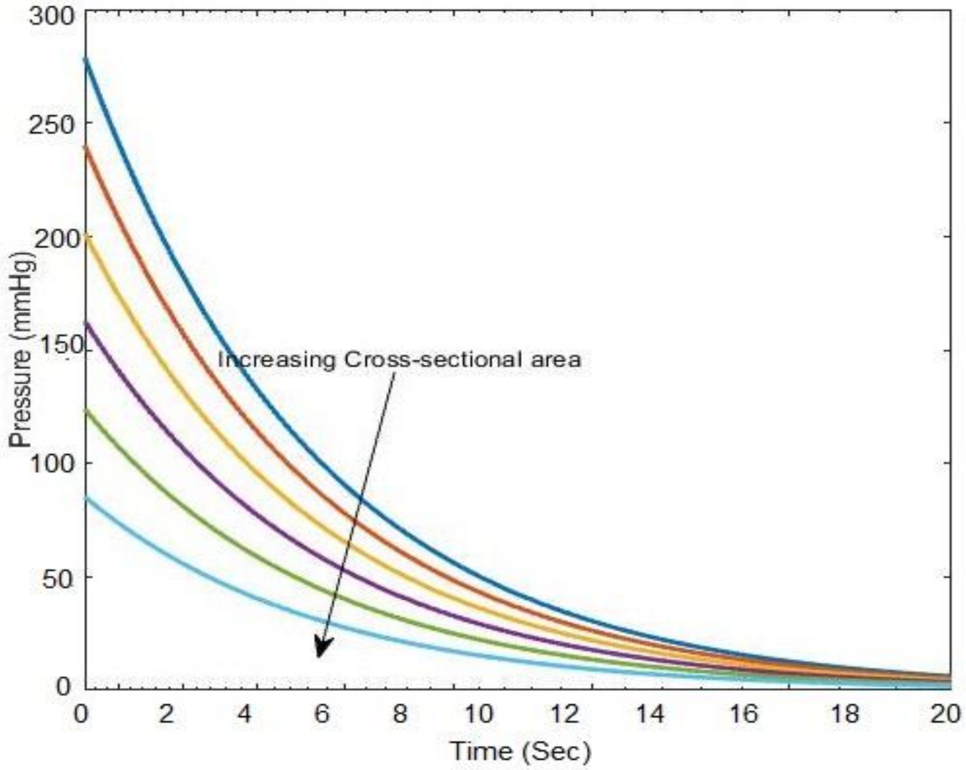


**Figure 4: Blood pressure for different cross-sectional areas of vessels.**

Moreover, the discrepancy between these two pressure sensory points exhibits considerable variation with vessel length. This finding aligns with the trends illustrated in Figure 7. In healthy adults, the systolic blood pressure typically ranges from 95 to 140 mm Hg, with an average value of approximately 120 mm Hg.

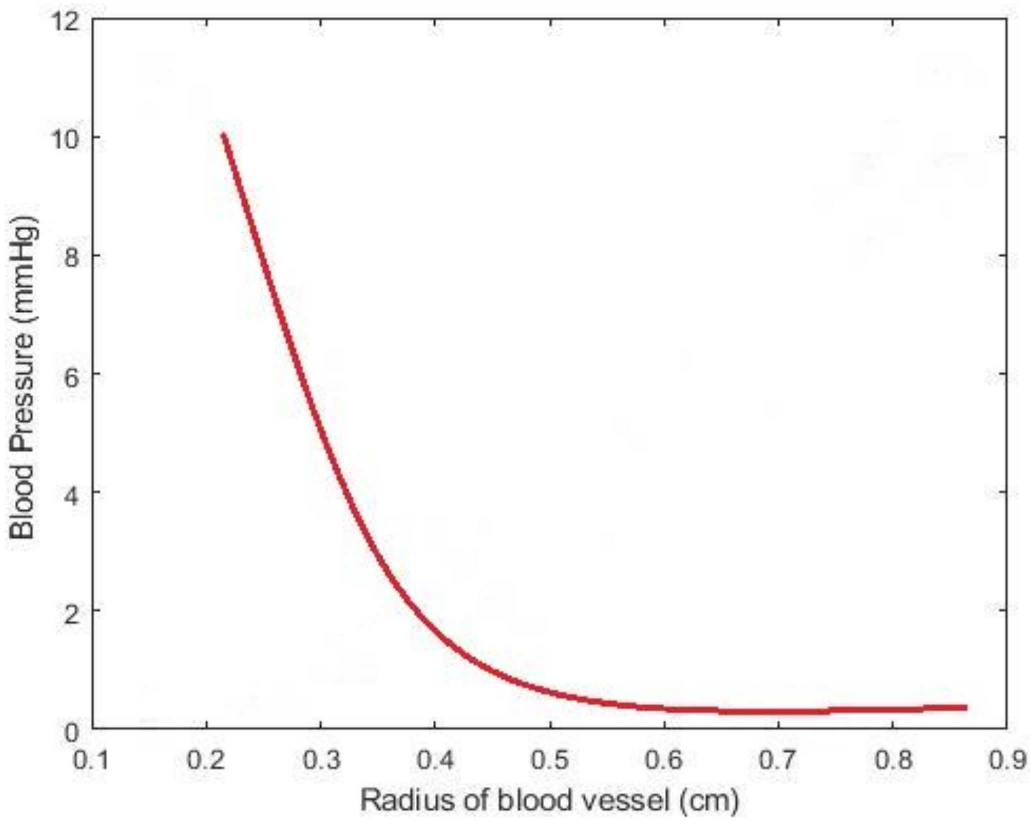


**Figure 5: Blood pressure with vessel radius**

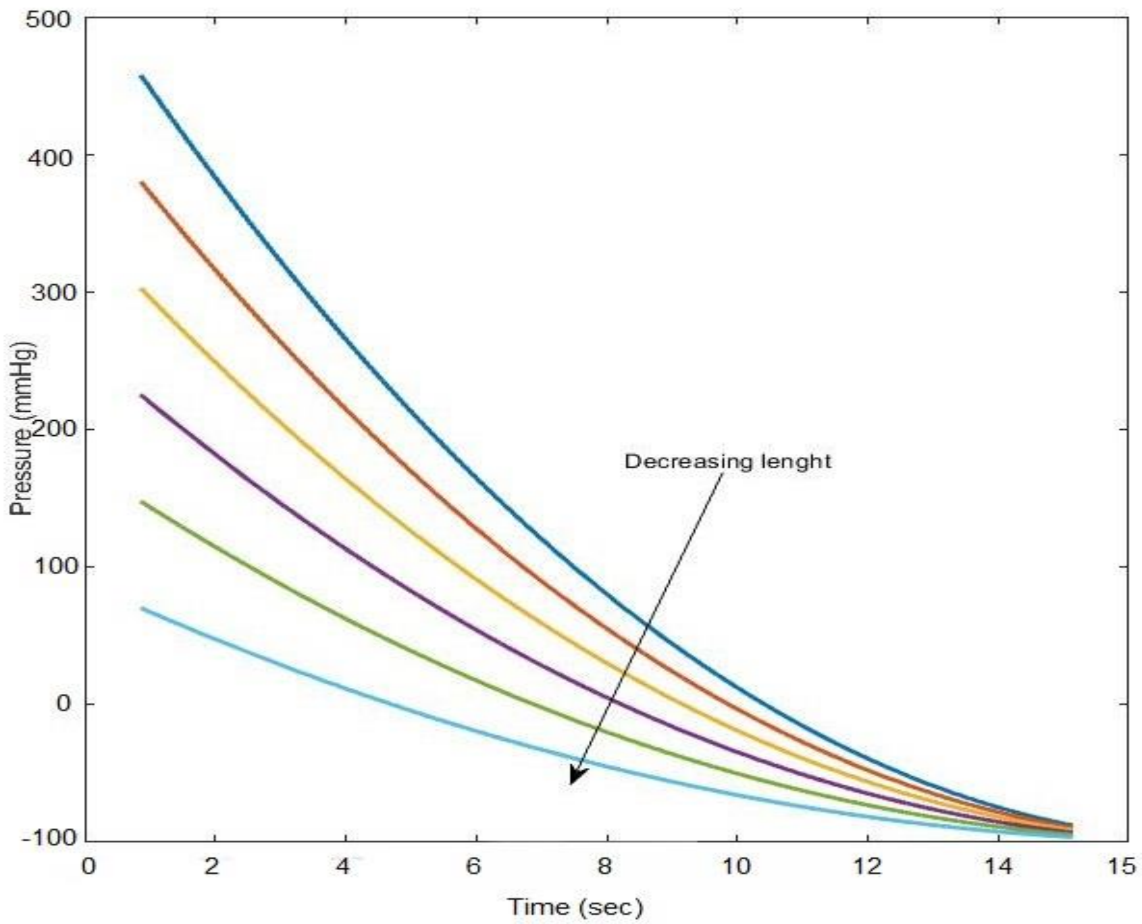


**Figure 6: Blood pressure for different vessel lengths**

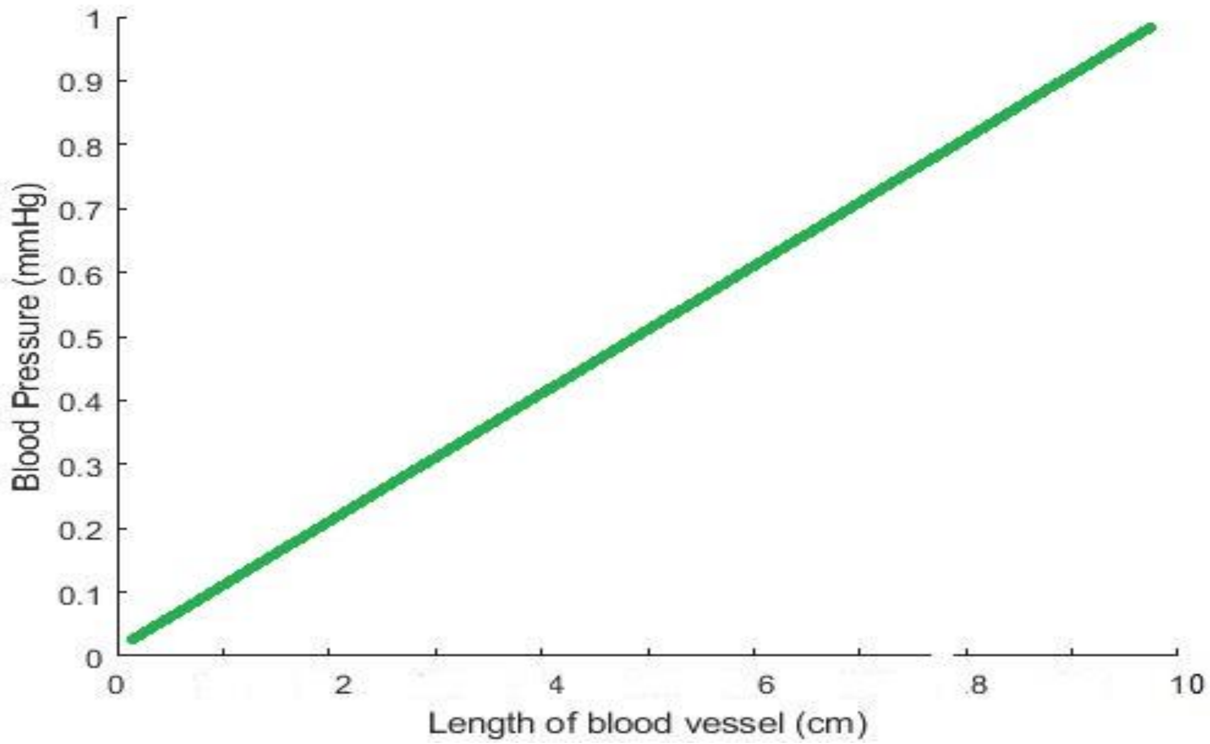


**Figure 7: Variation in pressure with vessel length**

However, these figures vary due to factors such as age, dietary habits, climate, and other environmental influences. Conversely, normal diastolic blood pressure typically ranges from 60 to 90 mm Hg, with an average value of approximately 80 mm Hg. These pressure readings are conventionally measured in arteries located in hand muscles. Fig. 4, 6, demonstrate that the range of pressures observed in the model aligns with the normal physiological values expected for an artery, taking length in to consideration [19]. Mathematical modeling is a robust methodology for investigating the dynamics of blood flow within the cardiovascular system. Through the application of mathematical constructs and the integration of pertinent physiological principles, researchers can gain valuable insights into the intricate behavior and functionality of the cardiovascular system. Continued research endeavors in this domain hold significant promise for advancing our understanding of cardiovascular physiology and fostering the development of innovative diagnostic and therapeutic interventions for cardiovascular diseases.

## Conclusion:

Simulation studies conducted based on the developed mathematical model offer valuable insights into the behavior of blood flow under diverse conditions. These simulations enable researchers to explore the effects of varying parameters on blood flow dynamics and pressure regulation. Through comparisons between simulated results and experimental data, researchers can ascertain the accuracy and predictive capabilities of mathematical models. An emerging area with promising applications in therapeutic simulation and medical image analysis involves the creation of computerized models depicting human organs. This study specifically focused on developing a mathematical model to depict blood flow within the cardiovascular system. Initially, the model considered only a few internal parameters; however, there are opportunities for enhancement by incorporating additional anatomical structures such as valves, precise heart chamber sizes, and refining the constitutive law to achieve a more realistic representation. It is essential to clarify that the primary aim of the study was not to construct the most intricate and detailed heart model. Instead, the aim is to study the model's complexity and to investigate the effects of alterations in cross-sectional area, pressure gradient, and blood channel length on both blood pressure and flow rate. Despite the simplifications introduced through various assumptions, the model remains valid because it effectively illustrates how changes in pressure gradients and cross-sectional area influence the flow rate and how the length and cross-sectional area of vessels impact blood pressure.

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